

# Global existence and uniform boundedness in advective Lotka–Volterra competition system with nonlinear diffusion

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## Abstract

This paper investigates reaction–advection–diffusion systems with Lotka–Volterra dynamics subject to homogeneous Neumann boundary conditions. Under conditions on growth rates of the density–dependent diffusion and sensitivity functions, we prove the global existence of classical solutions to the system and show that they are uniformly bounded in time. We also obtain the global existence and uniform boundedness for the corresponding parabolic–elliptic systems. Our results suggest that attraction (positive taxis) inhibits blowups in Lotka–Volterra competition systems.

**Keywords:** Lotka–Volterra competition system, nonlinear diffusion, global existence, boundedness

## 1 Introduction

This paper is concerned with the global existence and boundedness of  $(u, v) = (u(x, t), v(x, t))$  to reaction–advection–diffusion systems of the following form

$$\begin{cases} u_t = \nabla \cdot (D_1(u) \nabla u + \chi \phi(u) \nabla v) + (a_1 - b_1 u^\alpha - c_1 v)u, & x \in \Omega, t > 0, \\ v_t = D_2 \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \quad (1.1)$$

Here  $\Omega$  is a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 1$  and its smooth boundary  $\partial\Omega$  is endowed with unit outer normal  $\mathbf{n}$ .  $a_i$ ,  $b_i$ ,  $c_i$ ,  $i = 1, 2$ ,  $D_2$  and  $\chi$  are positive constants, while  $D_1$  and  $\phi$  are  $C^2$ –smooth functions of  $u$ . We assume there exist some positive constants  $M_i$ ,  $m_i > 0$ ,  $i = 1, 2$  such that

$$D_1(u) \geq M_1(1 + u)^{m_1}, \forall u \geq 0, \quad (1.2)$$

and

$$0 \leq \phi(u) \leq M_2 u^{m_2}, \forall u \geq 0. \quad (1.3)$$

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System (1.1) can be used to model the evolution of population distributions of two competing species subject to Lotka–Volterra dynamics. Consider two species with population densities at space–time location  $(x, t) \in \Omega \times \mathbb{R}^+$  denoted by  $u(x, t)$  and  $v(x, t)$ , respectively. Diffusions describe the random dispersals of the species as an anti–crowding mechanism and they are taken to be spatially local and against the direction of population gradient of the focal species. Moreover such anti–crowding motion changes with respect to the variation of the population density, and therefore we assume that  $D_1$  is a function of  $u$ , while  $D_2$  is chosen to be a positive constant for the simplicity of our analysis. The advection  $\chi\phi(u)\nabla v$ , or the cross–diffusion, accounts for the directed dispersal due to the population pressure from competing species  $v$ , and it is along with the direction of population gradient  $\nabla v$ . In (1.1) the function  $\phi(u)$  interprets the variation of the advection intensity with respect to population density  $u$ . The population kinetics are assumed to be of Lotka–Volterra type.

The initial step to understand the spatial–temporal dynamics of (1.1) is to study its global well–posedness. When the domain  $\Omega$  is one–dimensional, by applying the standard parabolic maximum principle and Moser–Alikakos  $L^p$  iteration, one can easily prove the global existence and uniform boundedness of (1.1). It is the goal of this paper to investigate the effect of growth rates  $m_i$  and decay rate  $\alpha$ , although far from being well understood, on the global existence and uniform boundedness of the system over multi–dimensional domain. Our first main result reads as follows.

**Theorem 1.1.** *Suppose that  $\Omega \in \mathbb{R}^N$ ,  $N \geq 2$ , is a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 1$ . Assume that the smooth functions  $D_1(u)$  and  $\phi(u)$  satisfy (1.2) and (1.3) respectively with*

$$m_2 - m_1 < \begin{cases} \frac{2}{N}, & \text{if } 0 < \alpha < 1, \\ \frac{3N+2}{N(N+2)}, & \text{if } \alpha \geq 1, \end{cases} \quad (1.4)$$

*then for any nonnegative  $(u_0, v_0) \in C^\kappa(\bar{\Omega}) \times W^{1,\infty}(\Omega)$ ,  $\kappa > 1$ , there exists at least one couple  $(u, v)$  of nonnegative bounded functions each belonging to  $C^0(\bar{\Omega} \times [0, \infty)) \cap C^{2,1}(\bar{\Omega} \times (0, \infty))$  which solves (1.1) classically. Moreover if  $(u_0, v_0) \in W^{k,p}(\Omega) \times W^{k,p}(\Omega)$  for some  $k > 1$  and  $p > N$ , the bounded solution above is unique.*

By a different approach we are able to prove the following result under a condition different from (1.4).

**Theorem 1.2.** *Suppose that all conditions in Theorem 1.1 hold except that*

$$2m_2 - m_1 < \begin{cases} \max\{\alpha, m_1\} + \frac{2}{N}, & \text{if } 0 < \alpha < 1, \\ \max\{\alpha, m_1\} + \frac{4}{N+2}, & \text{if } \alpha \geq 1, \end{cases} \quad (1.5)$$

*then all the conclusions in Theorem 1.1 hold, i.e., the solutions to (1.1) are global and uniformly bounded in time.*

In the absence of advection with  $\chi = 0$ , and when  $D_1(u) \equiv D_1$ ,  $\alpha = 1$ , (1.1) reduces to the classical diffusive Lotka–Volterra competition model

$$\begin{cases} u_t = D_1 \Delta u + (a_1 - b_1 u - c_1 v)u, & x \in \Omega, t > 0, \\ v_t = D_2 \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \quad (1.6)$$

Thanks to the standard parabolic maximum principles, it is quite obvious that the solution  $(u, v)$  to (1.6) exists globally and is uniformly bounded [12, 13]. It is well known that its positive homogeneous solutions  $(\bar{u}, \bar{v})$  is the global (exponential) attractor of (1.6) in weak competition case  $\frac{b_1}{b_2} > \frac{a_1}{a_2} > \frac{c_1}{c_2}$  [12, 15], and (1.6) does not admit stable nonconstant steady states when  $\Omega$

is convex [20] or one of the diffusion rates  $D_i$  is large [15, 29]. On the other hand, the system admits nonconstant positive steady states when  $\Omega$  is non-convex (e.g. of dumb-bell shaped) in the strong competition case  $\frac{b_1}{b_2} < \frac{a_1}{a_2} < \frac{c_1}{c_2}$ , with properly chosen (small) diffusion rates [36, 37, 38, 39]. See [29, 30, 51] for further discussions on (1.6).

Though it is not entirely unrealistic to assume that mutually interacting species disperse over the habitat purely randomly, from the viewpoint of mathematical modeling, it is interesting and important to incorporate advection or cross-diffusion into system (1.6), which accounts for the dispersal pressure due to population gradient of the intra- and/or inter-species. On the other hand, one of the most interesting phenomena in ecological evolutions is the well observed segregation of competition species, i.e., some regions of the habitat are dominated by one species and the rest by the other, however in most cases, system (1.6) inhibits the formations of nontrivial patterns such as boundary spikes, transition layers etc., which can be used to model the aforementioned segregation. For this purpose, the following model with advection was proposed and studied in [51]

$$\begin{cases} u_t = \nabla \cdot (D_1 \nabla u + \chi u \nabla v) + (a_1 - b_1 u - c_1 v)u, & x \in \Omega, t > 0, \\ v_t = D_2 \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \quad (1.7)$$

where all the parameters are positive constants. In [51], global existence and boundedness are obtained for this fully parabolic system when  $\Omega$  is one-dimensional and for its parabolic-elliptic counterpart when  $\Omega$  is multi-dimensional and  $\frac{\chi}{D_2}$  is small. Steady state bifurcation is performed to establish the existence and stability of its nonconstant stationary solutions. Moreover, it is shown that (1.7) admits transition-layer steady states when  $\chi$  and  $1/D_2$  are sufficiently large. Kuto and Tsujikawa have done a nice parallel work [23] on the limiting structure of stationary solutions to (1.7) as diffusion and advection of one of the species tend to infinity. Moreover, they investigated internal and boundary layer steady states of the limiting system. These nonconstant steady states can be used to model the aforementioned segregation phenomenon. Recently it is proved in [41] that extinction through competition does not occur in (1.7) in the weak competition case provided with small initial data. Global existence and nonconstant steady states of (1.1) with sublinear sensitivity are obtained in [52] when  $\Omega$  is a multi-dimensional bounded domain.

In this work, we extend model (1.7) to the more realistic (1.1) with nonconstant diffusion by assuming that the random dispersal rate of species depends nonlinearly on the population density of the focal species  $u$ . Moreover, the density-dependent sensitivity means that the advective velocity of species  $u$  varies with different population density. By nonlinear diffusion and sensitivity, we are able to use (1.1) to describe population-induced dispersals in ecological applications. Here for the simplicity of our analysis  $D_2$  is assumed to be a positive constant and we shall focus the interplay between  $m_i$  and  $\alpha$  on our global existence results.

We would like to mention that (1.1) serves as a prototype for reaction-diffusion systems with cross-diffusion which models population pressures created by the competitions. For example, Shigesada, Kawasaki and Teramoto [42] proposed the following system in 1979 to model the segregation phenomenon of two competing species

$$\begin{cases} u_t = \Delta[(d_1 + \rho_{11}u + \rho_{12}v)u] + (a_1 - b_1u - c_1v)u, & x \in \Omega, t > 0, \\ v_t = \Delta[(d_2 + \rho_{21}u + \rho_{22}v)v] + (a_2 - b_2u - c_2v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \quad (1.8)$$

which takes into consideration both *self-diffusions*  $\rho_{11}, \rho_{22}$  and *cross-diffusions*  $\rho_{12}, \rho_{21}$ . (1.8) has received adequate attention over the past few decades since its appearance, and a great deal of effort has been devoted to studying its global existence [9, 10, 28, 31, 43, 47, 48, 59] and

positive steady states [21, 22, 23, 29, 30, 32, 33, 40, 50, 58]. To compare (1.8) with (1.1), we let  $\rho_{21} = \rho_{22} = 0$  and rewrite it into the following form

$$\begin{cases} u_t = \nabla \cdot [(d_1 + 2\rho_{11}u + \rho_{12}v)\nabla u + \rho_{12}u\nabla v] + (a_1 - b_1u - c_1v)u, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v + (a_2 - b_2u - c_2v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega. \end{cases} \quad (1.9)$$

which is a special case of (1.1) with  $m_2 = m_1 = \alpha = 1$ . It is proved in [31] that when space dimension  $N = 2$ , if  $u_0, v_0 \in W^{k,p}$  for some  $k > N$ , then (1.9) has a unique global solution which solves the system classically. This global existence result can be rediscovered by both Theorem 1.1 and Theorem 2.1 since both (1.4) and (1.5) obviously hold. Moreover our results show that the global solutions are uniformly bounded in time which was not available in [31].

Another example is the following model proposed in [5, 6] to study the dispersal strategies leading to ideal free distribution of populations in evolutionary ecology

$$\begin{cases} u_t = \nabla \cdot (d_1\nabla u - \chi u\nabla(m - u - v)) + (m - u - v)u, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v + (m - u - v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases} \quad (1.10)$$

where  $d_1, d_2$  and  $\chi$  are positive constants.  $m = m(x) \in C^{2+\gamma}(\bar{\Omega})$  and  $m(x) > 0$  in  $\Omega$ . Lou *et al.* [35] studied the bounded classical global solutions to the following system over multi-dimensional domain  $N \geq 1$ . We note that (1.10) can be rewritten as

$$\begin{cases} u_t = \nabla \cdot ((d_1 + \chi u)\nabla u + \chi u\nabla v - \chi u\nabla m) + (m - u - v)u, & x \in \Omega, t > 0, \\ v_t = d_2\Delta v + r(m - u - v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, v(x, 0) = v_0(x) \geq 0, & x \in \Omega, \end{cases}$$

hence it is a special case of (1.1) with  $m_1 = m_2 = \alpha = 1$  and the global well-posedness follows from Theorem 1.1 or Theorem 1.2. We refer to [7, 8, 14, 16, 17, 19, 25, 26, 27, 34, 44] and the references therein for works cross-diffusion systems.

We would like to mention that (1.1) is very similar as the nonlinear diffusion Keller–Segel models of chemotaxis, which describes the directed movements of cellular organisms in response to chemical stimulus. In particular, the chemotaxis is positive if the cells move towards high concentration of attractive chemical (sugar, nutrition e.g.) and chemotaxis is negative if the cells move against repulsive chemical (poison, hazardous materials e.g.). It is also necessary to point out that the logistic growth in Lotka–Volterra dynamics, which inhibits solutions from blowing up in finite or infinite time for purely diffusive models, might not be sufficient to prevent blowups when advection or chemotaxis is present. For example, Le and Nguyen [29] gave an example of finite-time blowup solutions to a cross-diffusion system subject to Lotka–Volterra dynamics. See [24, 56, 57] for counter-examples for chemotaxis models with logistic growth. Moreover, we refer the reader to [3, 4, 11, 18, 45, 46, 49, 53, 60, 61, 62] for works on chemotaxis models with nonlinear diffusions.

The rest part of this paper is organized as follows. In Section 2, we present the existence and an extension criterion of local-in-time solutions to (1.1) together with their important properties. In Section 3, we establish several *a priori* estimates which are essential for the proof of Theorem 1.1 and Theorem 1.2. Finally, we study the parabolic–elliptic system of (1.1) in Section 4. For parabolic–elliptic system with repulsion, we prove its global existence and boundedness in Theorem 4.1 and Theorem 4.2 under conditions on  $m_i$  and  $\alpha$  milder than (1.4) and (1.5); moreover for parabolic–elliptic with attraction (i.e., change  $\chi$  to  $-\chi$ ), we prove in Theorem 4.3 that the solutions are global and bounded for as long as one of  $m_1, m_2$  and  $\alpha$

is nonnegative. Our results indicate that repulsion or negative chemotaxis, which acts as a smoothing process for Keller–Segel models, destabilizes the spatially homogeneous solution of Lotka–Volterra competition systems (see Proposition 1 in [51] e.g.).

In the sequel, we denote  $C_{ix}/C_{ixx}$  as the  $x$ -th/ $xx$ -th positive constant in the  $i$ -th section.

## 2 Local existence and preliminary results

The mathematical analysis of global well-posedness of (1.1) is delicate since maximum principle does not apply for the  $u$  equation. We first study the local well-posedness of (1.1) following the fundamental theory developed by Amann [2].

**Proposition 1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 1$ . Let  $a_i, b_i, c_i, \alpha, D_2$  be positive and suppose that  $D_1(u)$  and  $\phi(u)$  are  $C^2$  smooth functions and they satisfy (1.2) and (1.3) for positive constants  $m_i$  and  $M_i$ ,  $i = 1, 2$ . Assume that for some  $\kappa > 1$  and  $p > N$ ,  $(u_0, v_0)$  belongs to  $(W^{\kappa,p}(\Omega))^2$  and  $u_0, v_0 \geq 0$ ,  $\neq 0$  in  $\bar{\Omega}$ . Then there exist  $T_{\max} \in (0, \infty]$  and a unique couple  $(u, v)$  of nonnegative functions from  $C^0(\bar{\Omega} \times [0, T_{\max})) \cap C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$  solving (1.1) classically in  $\Omega \times (0, T_{\max})$ . Moreover  $u(x, t) \geq 0$  and  $v(x, t) \geq 0$  in  $\Omega \times (0, T_{\max})$  and the following dichotomy holds:*

$$\text{either } T_{\max} = \infty \quad \text{or} \quad \limsup_{t \nearrow T_{\max}^-} \|u(\cdot, t)\|_{L^\infty(\Omega)} = \infty. \quad (2.1)$$

Next we collect some properties of the local solution.

**Lemma 2.1.** *Let  $(u, v)$  be a nonnegative classical solution of (1.1) in  $\Omega \times (0, T_{\max})$ . Then the following statements hold true:*

(i) *there exists a positive constant  $C$  such that*

$$\int_{\Omega} u(x, t) dx \leq C, \forall t \in (0, T_{\max}) \quad (2.2)$$

and

$$0 \leq v(x, t) \leq \max \left\{ \frac{a_2}{c_2}, \|v_0\|_{L^\infty(\Omega)} \right\}, \forall (x, t) \in \Omega \times (0, T_{\max}); \quad (2.3)$$

(ii) *for each  $s \in [1, \frac{N}{N-1})$ , there exists  $C_s > 0$  such that*

$$\|v(\cdot, t)\|_{W^{1,s}(\Omega)} \leq C_s, \forall t \in (0, T_{\max}); \quad (2.4)$$

moreover if  $u \in L^p(\Omega)$  for some  $p \in [1, \infty)$ , there exists a positive constant  $C$  dependent on  $\|v_0\|_{L^q(\Omega)}$  and  $|\Omega|$  such that

$$\|v(\cdot, t)\|_{W^{1,q}(\Omega)} \leq C \left( 1 + \sup_{s \in (0, t)} \|u(\cdot, s)\|_{L^p(\Omega)} \right), \forall t \geq 0, \quad (2.5)$$

where  $q \in [1, \frac{Np}{N-p})$  if  $p \in [1, N)$ ,  $q \in [1, \infty)$  if  $p = N$  and  $q = \infty$  if  $p > N$ .

*Proof.* First of all, the nonnegativity of  $u(x, t)$  and (2.3) follows if we apply the standard parabolic maximum principles and Hopf's lemma to the  $v$ -equation. To show (2.2), we integrate the  $u$ -equation in (1.1) over  $\Omega$  to get

$$\frac{d}{dt} \int_{\Omega} u = a_1 \int_{\Omega} u - b_1 \int_{\Omega} u^{\alpha+1} - c_1 \int_{\Omega} uv \leq a_1 \int_{\Omega} u - b_1 \int_{\Omega} u^{\alpha+1}. \quad (2.6)$$

After applying the Young's inequality  $(a_1 + 1) \int_{\Omega} u \leq b_1 \int_{\Omega} u^{\alpha+1} + C_{\Omega}$  for some positive constant  $C_{\Omega}$ , we obtain from (2.6)

$$\frac{d}{dt} \int_{\Omega} u + \int_{\Omega} u \leq C$$

and solving this differential inequality by Grönwall's lemma leads us to (2.2).

To verify (ii), we observe that (2.4) is a special case of (2.5) with  $p = 1$  and therefore we shall only prove the latter. To this end, we write the following abstract formula of  $v$

$$v(\cdot, t) = e^{D_2(\Delta-1)t}v_0 + \int_0^t e^{D_2(\Delta-1)(t-s)}(D_2v(\cdot, s) + g(u(\cdot, s), v(\cdot, s)))ds, \quad (2.7)$$

where  $g(u, v) = (a_2 - b_2u - c_2v)v$ . Thanks to the  $L^p$ - $L^q$  estimates between semigroups  $\{e^{t\Delta}\}_{t \geq 0}$  (Lemma 1.3 of [54] e.g.), we can find positive constants  $C_{21}$ ,  $C_{22}$  and  $C_{23}$  such that

$$\begin{aligned} & \|v(\cdot, t)\|_{W^{1,q}} \\ &= \left\| e^{D_2(\Delta-1)t}v_0 + \int_0^t e^{D_2(\Delta-1)(t-s)}(D_2v(\cdot, s) + g(u(\cdot, s), v(\cdot, s)))ds \right\|_{W^{1,q}} \\ &\leq C_{21}\|v_0\|_{L^p} + C_{21} \int_0^t e^{-D_2\nu(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})(\|u(\cdot, t)\|_{L^p} + 1)ds \\ &\leq C_{22} + C_{23} \int_0^t e^{-D_2\nu(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})\|u(\cdot, s)\|_{L^p}ds \\ &\leq C_{22} + C_{23} \left( \int_0^t e^{-D_2\nu(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})ds \right) \sup_{s \in (0,t)} \|u(\cdot, s)\|_{L^p}, \end{aligned} \quad (2.8)$$

where  $\nu$  is the first Neumann eigenvalue of  $-\Delta$ . On the other hand, under the conditions on  $q$  after (2.5) we have

$$\sup_{t \in (0, \infty)} \int_0^t e^{-D_2\nu(t-s)}(1 + (t-s)^{-\frac{1}{2}-\frac{N}{2}(\frac{1}{p}-\frac{1}{q})})ds < \infty,$$

and therefore (2.5) follows from (2.8).  $\square$

According to (2.4) in Lemma 2.1,  $\|\nabla v(\cdot, t)\|_{L^s}$  is bounded for  $s \in [1, \frac{N}{N-1})$ . Therefore for  $N = 1$ , one has the boundedness of  $\|\nabla v(\cdot, t)\|_{L^s}$  for each fixed  $s \in [1, \infty)$ . By the standard Moser–Alikakos iteration we can easily prove the global existence and boundedness in Theorem 1.1 and Theorem 1.2 for  $N = 1$ . Therefore, in the sequel we shall focus on  $N \geq 2$  for which one has the boundedness of  $\|\nabla v(\cdot, t)\|_{L^s}$  for each fixed  $s \in [1, \frac{N}{N-1})$ . In this case, we want to point out that  $\frac{N}{N-1} \leq 2$  and our next result indicates that  $s = 2$  can be achieved if  $\alpha \geq 1$ .

**Lemma 2.2.** *Suppose that  $\alpha \geq 1$ , then there exists a positive constant  $C$  such that*

$$\|\nabla v(\cdot, t)\|_{L^2(\Omega)} \leq C, \forall t \in (0, T_{\max}). \quad (2.9)$$

*Proof.* Testing the  $v$ -equation in (1.1) by  $\Delta v$  and then integrating it over  $\Omega$  by parts, we have

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla v|^2 &= \int_{\Omega} \nabla v \cdot \nabla v_t \\ &= \int_{\Omega} \nabla v \cdot \nabla [D_2\Delta v + (a_2 - b_2u - c_2v)v] \\ &= -D_2 \int_{\Omega} |\Delta v|^2 + a_2 \int_{\Omega} |\nabla v|^2 + \int_{\Omega} b_2uv\Delta v - 2c_2 \int_{\Omega} v|\nabla v|^2 \\ &\leq -D_2 \int_{\Omega} |\Delta v|^2 + a_2 \int_{\Omega} |\nabla v|^2 + \frac{b_2^2}{2D_2} \int_{\Omega} u^2v^2 + \frac{D_2}{2} \int_{\Omega} |\Delta v|^2 \\ &\leq -\frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + a_2 \int_{\Omega} |\nabla v|^2 + \mu \int_{\Omega} u^2, \end{aligned} \quad (2.10)$$

where  $\mu := \frac{b_2^2 \|v\|_{L^\infty(\Omega)}^2}{2D_2}$  and  $C_{24}$  is a positive constant. By Sobolev interpolation inequality and in light of the boundedness of  $\|v\|_{L^\infty(\Omega)}$ , we obtain that for positive constants  $C_{25}$  and  $C_{26}$

$$\left(a_2 + \frac{1}{2}\right) \int_{\Omega} |\nabla v|^2 \leq \frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + C_{25} \int_{\Omega} v^2 \leq \frac{D_2}{2} \int_{\Omega} |\Delta v|^2 + C_{26}.$$

Multiplying (2.6) by  $\frac{2\mu}{b_1}$  and then adding it to (2.10), we have

$$\begin{aligned} & \frac{d}{dt} \left( \frac{2\mu}{b_1} \int_{\Omega} u + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \right) + \left( \frac{2\mu}{b_1} \int_{\Omega} u + \frac{1}{2} \int_{\Omega} |\nabla v|^2 \right) \\ & \leq \left( \frac{2a_1\mu}{b_1} \int_{\Omega} u - \mu \int_{\Omega} u^{\alpha+1} \right) + \mu \left( \int_{\Omega} u^2 - \int_{\Omega} u^{\alpha+1} \right) + C_{27} \leq C_{28}, \end{aligned}$$

where  $C_{27}$  and  $C_{28}$  are positive constant, and therefore  $\|\nabla v\|_{L^2}$  is bounded for all  $t \in (0, T_{\max})$  as desired.  $\square$

### 3 Parabolic–parabolic system in multi–dimensional domain

According to Proposition 1 and (2.3), in order to prove Theorem 1.1 and Theorem 1.2, it is sufficient to show that  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$  is bounded for each time  $t \in (0, T_{\max})$  and therefore  $T_{\max} = \infty$  and the solution is global. Indeed we will show that  $\|u(\cdot, t)\|_{L^\infty(\Omega)}$  is uniformly bounded in  $t \in (0, \infty)$ . To this end, it is sufficient to prove that  $\|u(\cdot, t)\|_{L^p(\Omega)}$  is bounded for some  $p$  large according to (2.5). For this purpose we will give a combined estimate on  $\int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q}$  for both  $p$  and  $q$  large based on the idea recently developed in [35, 46, 55] etc.

#### 3.1 A priori estimates

For any  $p \geq 2$ , we multiply the  $u$ -equation in (1.1) by  $u^{p-1}$  and then integrate it over  $\Omega$  by parts

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &= \int_{\Omega} u^{p-1} \nabla \cdot (D_1(u) \nabla u) + \int_{\Omega} u^{p-1} \nabla \cdot (\chi \phi(u) \nabla v) + \int_{\Omega} u^p (a_1 - b_1 u^\alpha - c_1 v) \\ &= -(p-1) \int_{\Omega} D_1(u) u^{p-2} |\nabla u|^2 - (p-1) \int_{\Omega} \chi \phi(u) u^{p-2} \nabla u \nabla v \\ &\quad + \int_{\Omega} u^p (a_1 - b_1 u^\alpha - c_1 v). \end{aligned} \tag{3.1}$$

In light of  $D_1(u) \geq M_1(1+u)^{m_1} > M_1 u^{m_1}$ , we have

$$\begin{aligned} (p-1) \int_{\Omega} D_1(u) u^{p-2} |\nabla u|^2 &\geq M_1 (p-1) \int_{\Omega} u^{p+m_1-2} |\nabla u|^2 \\ &= \frac{4M_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{\frac{p+m_1}{2}}|^2, \end{aligned} \tag{3.2}$$

where the identity follows from

$$u^{p+m_1-2} |\nabla u|^2 = \frac{4}{(p+m_1)^2} |\nabla u^{\frac{p+m_1}{2}}|^2.$$

Moreover Young's inequality implies

$$\begin{aligned} & -(p-1) \int_{\Omega} \chi \phi(u) u^{p-2} \nabla u \nabla v \\ & \leq \frac{M_1(p-1)}{2} \int_{\Omega} u^{p+m_1-2} |\nabla u|^2 + \frac{\chi^2(p-1)}{2M_1} \int_{\Omega} u^{p-m_1-2} \phi^2(u) |\nabla v|^2 \\ & \leq \frac{2M_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{\frac{p+m_1}{2}}|^2 + \frac{\chi^2 M_2^2(p-1)}{2M_1} \int_{\Omega} u^{p-m_1+2m_2-2} |\nabla v|^2 \end{aligned} \tag{3.3}$$

and

$$\left(a_1 + \frac{1}{p}\right) \int_{\Omega} u^p \leq \frac{b_1}{2} \int_{\Omega} u^{p+\alpha} + C_{31}, \quad (3.4)$$

where  $C_{31}$  is a positive constant dependent on  $p$ . Thanks to (3.2)–(3.4) we have from (3.1)

$$\begin{aligned} & \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p + \frac{1}{p} \int_{\Omega} u^p + \frac{2M_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{\frac{p+m_1}{2}}|^2 + \frac{b_1}{2} \int_{\Omega} u^{p+\alpha} \\ & \leq \frac{\chi^2 M_2^2 (p-1)}{2M_1} \int_{\Omega} u^{p-m_1+2m_2-2} |\nabla v|^2 + C_{31}. \end{aligned} \quad (3.5)$$

On the other hand, for any  $q > 1$ , we have from the  $v$ -equation in (1.1)

$$\begin{aligned} & \frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} = \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla v_t \\ & = \overbrace{D_2 \int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla \Delta v}^{I_1} + \overbrace{\int_{\Omega} |\nabla v|^{2q-2} \nabla v \cdot \nabla [(a_2 - b_2 u - c_2 v)v]}^{I_2}. \end{aligned} \quad (3.6)$$

In light of the identity

$$\nabla v \cdot \nabla \Delta v = \frac{1}{2} \Delta |\nabla v|^2 - |D^2 v|^2,$$

we first estimate  $I_1$  in (3.6) through

$$\begin{aligned} I_1 &= \frac{D_2}{2} \int_{\Omega} |\nabla v|^{2q-2} \Delta |\nabla v|^2 - D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 \\ &= \frac{D_2}{2} \int_{\partial\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n} - \frac{D_2}{2} \int_{\Omega} \nabla |\nabla v|^{2q-2} \cdot \nabla |\nabla v|^2 - D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 \\ &= \overbrace{\frac{D_2}{2} \int_{\partial\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n}}^{I_{11}} - \overbrace{\frac{(q-1)D_2}{2} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2}^{I_{12}} \\ &\quad - \overbrace{D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2}^{I_{13}}. \end{aligned} \quad (3.7)$$

To estimate  $I_{11}$ , we invoke the inequality  $\frac{\partial |\nabla v|^2}{\partial n} \leq C_{\Omega} |\nabla v|^2$  (e.g. inequality (2.4) in [18]) with  $C_{\Omega}$  being a positive constant depending only on the curvatures of  $\partial\Omega$  to deduce

$$I_{11} = \frac{D_2}{2} \int_{\partial\Omega} |\nabla v|^{2q-2} \frac{\partial |\nabla v|^2}{\partial n} \leq \frac{D_2 C_{\Omega}}{2} \int_{\partial\Omega} |\nabla v|^{2q} := \bar{C}_{\Omega} \left\| |\nabla v|^q \right\|_{L^2(\partial\Omega)}^2. \quad (3.8)$$

By taking  $r \in (0, \frac{1}{2})$ , we have from (1.9) in [18] that the embedding  $W^{r+\frac{1}{2},2}(\Omega) (\hookrightarrow W^{r,2}(\partial\Omega)) \hookrightarrow L^2(\partial\Omega)$  is compact and therefore there exists a positive constant  $C_{32}$  such that

$$\left\| |\nabla v|^q \right\|_{L^2(\partial\Omega)} \leq C_{32} \left\| |\nabla v|^q \right\|_{W^{r+\frac{1}{2},2}(\Omega)}. \quad (3.9)$$

Let  $h_1 \in (0, 1)$  satisfy

$$\frac{1}{2} - \frac{r+\frac{1}{2}}{N} = \left(1 - h_1\right) \frac{q}{s} + h_1 \left(\frac{1}{2} - \frac{1}{N}\right),$$

or

$$h_1 = \frac{\frac{q}{s} - \left(\frac{1}{2} - \frac{1}{2N} - \frac{r}{N}\right)}{\frac{q}{s} - \left(\frac{1}{2} - \frac{1}{N}\right)} \in \left(r + \frac{1}{2}, 1\right),$$



where we choose  $s \in [1, \frac{N}{N-1})$  if  $\alpha < 1$  and  $s = 2$  if  $\alpha \geq 1$ , then we can invoke the fractional Gagliardo–Nirenberg interpolation inequality to deduce

$$\begin{aligned} \left\| |\nabla v|^q \right\|_{W^{r+\frac{1}{2},2}(\Omega)} &\leq C_{33} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{h_1} \left\| |\nabla v|^q \right\|_{L^{\frac{q}{q-1}}(\Omega)}^{1-h_1} + C_{34} \left\| |\nabla v|^q \right\|_{L^{\frac{q}{q-1}}(\Omega)} \\ &\leq C_{35} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{h_1} + C_{36}, \end{aligned} \quad (3.10)$$

where we have applied the fact that  $\|\nabla v\|_{L^s}$  is uniformly bounded. In conjunction with (3.9) and (3.10), we obtain from (3.8) through Young's inequality that

$$\begin{aligned} I_{11} &\leq 2\bar{C}_\Omega C_{32}^2 (C_{35}^2 \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{2h_1} + C_{36}^2) \\ &\leq \frac{(q-1)D_2}{q^2} \int_\Omega \left| \nabla |\nabla v|^q \right|^2 + C_{37} \end{aligned} \quad (3.11)$$

since  $h_1 < 1$ , where  $C_{37}$  is a positive constant. To estimate  $I_{12}$  we note

$$|\nabla v|^{2q-4} \left| \nabla |\nabla v|^2 \right|^2 = \frac{4}{q^2} \left| \nabla |\nabla v|^q \right|^2,$$

then

$$I_{12} = \frac{2(q-1)D_2}{q^2} \int_\Omega \left| \nabla |\nabla v|^q \right|^2. \quad (3.12)$$

Substituting (3.11) and (3.12) into (3.7) gives us

$$I_1 \leq -\frac{(q-1)D_2}{q^2} \int_\Omega \left| \nabla |\nabla v|^q \right|^2 - D_2 \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 + C_{38}. \quad (3.13)$$

To estimate  $I_2$ , we obtain from the integration by parts

$$\begin{aligned} I_2 &= \int_\Omega |\nabla v|^{2q-2} \nabla v \cdot \nabla [(a_2 - b_2 u - c_2 v)v] \\ &= - \int_\Omega (a_2 - b_2 u - c_2 v)v \nabla \cdot (|\nabla v|^{2q-2} \nabla v) \\ &= - \int_\Omega (a_2 - b_2 u - c_2 v)v |\nabla v|^{2q-2} \Delta v \\ &\quad - (q-1) \int_\Omega (a_2 - b_2 u - c_2 v)v |\nabla v|^{2q-4} \nabla |\nabla v|^2 \cdot \nabla v \\ &= - \overbrace{\int_\Omega (a_2 - c_2 v)v |\nabla v|^{2q-2} \Delta v}^{I_{21}} - \overbrace{(q-1) \int_\Omega (a_2 - c_2 v)v |\nabla v|^{2q-4} \nabla |\nabla v|^2 \cdot \nabla v}^{I_{22}} \\ &\quad + \overbrace{b_2 \int_\Omega uv |\nabla v|^{2q-2} \Delta v}^{I_{23}} + \overbrace{(q-1)b_2 \int_\Omega uv |\nabla v|^{2q-4} \nabla |\nabla v|^2 \cdot \nabla v}^{I_{24}}. \end{aligned} \quad (3.14)$$

We apply Young's inequality to have

$$\begin{aligned} -I_{21} &\leq \frac{D_2}{2N} \int_\Omega |\nabla v|^{2q-2} |\Delta v|^2 + \frac{N}{2D_2} \int_\Omega (a_2 - c_2 v)^2 v^2 |\nabla v|^{2q-2} \\ &\leq \frac{D_2}{2} \int_\Omega |\nabla v|^{2q-2} |D^2 v|^2 + C_{39} \int_\Omega |\nabla v|^{2q-2}, \end{aligned} \quad (3.15)$$

where  $C_{39}$  is a positive constant and the second inequality follows from the pointwise inequality  $|\Delta v|^2 \leq N|D^2 v|^2$ . Similarly we have

$$-I_{22} \leq \frac{(q-1)D_2}{16} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 + C_{310} \int_{\Omega} |\nabla v|^{2q-2}, \quad (3.16)$$

$$I_{23} \leq \frac{D_2}{2} \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + C_{311} \int_{\Omega} u^2 |\nabla v|^{2q-2}, \quad (3.17)$$

and

$$I_{24} \leq \frac{(q-1)D_2}{16} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 + C_{312} \int_{\Omega} u^2 |\nabla v|^{2q-2}, \quad (3.18)$$

for positive constants  $C_{310}$ ,  $C_{311}$  and  $C_{312}$ . Collecting (3.15)–(3.18), we infer from (3.14)

$$\begin{aligned} I_2 &\leq D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + \frac{(q-1)D_2}{8} \int_{\Omega} |\nabla v|^{2q-4} |\nabla |\nabla v|^2|^2 \\ &\quad + (C_{39} + C_{310}) \int_{\Omega} |\nabla v|^{2q-2} + (C_{311} + C_{312}) \int_{\Omega} u^2 |\nabla v|^{2q-2} \\ &= D_2 \int_{\Omega} |\nabla v|^{2q-2} |D^2 v|^2 + \frac{(q-1)D_2}{2q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ &\quad + (C_{39} + C_{310}) \int_{\Omega} |\nabla v|^{2q-2} + (C_{311} + C_{312}) \int_{\Omega} u^2 |\nabla v|^{2q-2}. \end{aligned} \quad (3.19)$$

Combining (3.19) with (3.13), we have from (3.6)

$$\begin{aligned} \frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} &\leq - \frac{(q-1)D_2}{2q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 + (C_{39} + C_{310}) \int_{\Omega} |\nabla v|^{2q-2} \\ &\quad + (C_{311} + C_{312}) \int_{\Omega} u^2 |\nabla v|^{2q-2} + C_{38} \end{aligned}$$

or equivalently

$$\begin{aligned} \frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} &+ \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} + \frac{(q-1)D_2}{2q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ &\leq C_{313} \int_{\Omega} |\nabla v|^{2q} + C_{314} \int_{\Omega} u^2 |\nabla v|^{2q-2} + C_{38}, \end{aligned} \quad (3.20)$$

where  $C_{313} = C_{39} + C_{310} + \frac{1}{2q}$  and  $C_{314} = C_{311} + C_{312}$ . Using Gagliardo–Nirenberg interpolation inequality and Young’s inequality we estimate

$$\begin{aligned} C_{313} \int_{\Omega} |\nabla v|^{2q} &= C_{313} \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^2 \\ &\leq C_{315} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{2h_2} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{2(1-h_2)} + C_{315} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^2 \\ &\leq \frac{(q-1)D_2}{4q^2} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^2 + C_{316}, \end{aligned} \quad (3.21)$$

where

$$h_2 = \frac{\frac{q}{s} - \frac{1}{2}}{\frac{q}{s} - (\frac{1}{2} - \frac{1}{N})} \in (0, 1),$$

and  $C_{316}$  depends on the boundedness of  $\left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^2 = \left\| |\nabla v| \right\|_{L^s(\Omega)}^{2q}$  due to (2.4) and (2.9).

In light of (3.20) and (3.21) we have

$$\begin{aligned} & \frac{1}{2q} \frac{d}{dt} \int_{\Omega} |\nabla v|^{2q} + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} + \frac{(q-1)D_2}{4q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ & \leq C_{314} \int_{\Omega} u^2 |\nabla v|^{2q-2} + C_{317}, \end{aligned} \quad (3.22)$$

where  $C_{317} = C_{38} + C_{316}$ . Finally by collecting (3.5) and (3.22) we conclude

$$\begin{aligned} & \frac{d}{dt} \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) + \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) \\ & + \frac{2M_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{\frac{p+m_1}{2}}|^2 + \frac{(q-1)D_2}{4q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 + \frac{b_1}{2} \int_{\Omega} u^{p+\alpha} \\ & \leq \frac{\chi^2 M_2^2 (p-1)}{2M_1} \overbrace{\int_{\Omega} u^{p-m_1+2m_2-2} |\nabla v|^2}^{I_{31}} + C_{314} \overbrace{\int_{\Omega} u^2 |\nabla v|^{2q-2}}^{I_{32}} + C_{317}. \end{aligned} \quad (3.23)$$

We are now ready to present the following a priori estimates.

**Lemma 3.1.** *Let  $(u, v)$  be a positive classical solution of (1.1) in  $\Omega \times (0, T_{\max})$ . Suppose that  $m_1$  and  $m_2$  satisfy condition (1.4). Then for large  $p$  and  $q$  there exists a positive constant  $C(p, q)$  such that*

$$\int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q} \leq C(p, q), \forall t \in (0, \infty). \quad (3.24)$$

*Proof.* Let  $\mu_i > 1$  be an arbitrary real number to be selected and  $\mu'_i := \frac{\mu_i}{\mu_i - 1}$  be its conjugate. We can apply Hölder's inequality to estimate  $I_{3i}$  in (3.23)

$$I_{31} \leq \left( \int_{\Omega} u^{(p-m_1+2m_2-2)\mu_1} \right)^{\frac{1}{\mu_1}} \cdot \left( \int_{\Omega} |\nabla v|^{2\mu'_1} \right)^{\frac{1}{\mu'_1}} := \left( \int_{\Omega} u^{\lambda_1 \mu_1} \right)^{\frac{1}{\mu_1}} \cdot \left( \int_{\Omega} |\nabla v|^{\kappa_1 \mu'_1} \right)^{\frac{1}{\mu'_1}}$$

and

$$I_{32} \leq \left( \int_{\Omega} u^{2\mu_2} \right)^{\frac{1}{\mu_2}} \cdot \left( \int_{\Omega} |\nabla v|^{2(q-1)\mu'_2} \right)^{\frac{1}{\mu'_2}} := \left( \int_{\Omega} u^{\lambda_2 \mu_2} \right)^{\frac{1}{\mu_2}} \cdot \left( \int_{\Omega} |\nabla v|^{\kappa_2 \mu'_2} \right)^{\frac{1}{\mu'_2}},$$

which can be simplified as

$$I_{3i} \leq \left( \int_{\Omega} u^{\lambda_i \mu_i} \right)^{\frac{1}{\mu_i}} \cdot \left( \int_{\Omega} |\nabla v|^{\kappa_i \mu'_i} \right)^{\frac{1}{\mu'_i}}, i = 1, 2, \quad (3.25)$$

where for consistency of notation we denote

$$\lambda_1 = p - m_1 + 2m_2 - 2, \lambda_2 = 2, \quad (3.26)$$

and

$$\kappa_1 = 2, \kappa_2 = 2(q-1). \quad (3.27)$$

By Gagliardo–Nirenberg interpolation inequality, there exist positive constants  $C_{318}$  and  $C_{319}$  such that in (3.25)

$$\begin{aligned} & \left( \int_{\Omega} u^{\lambda_i \mu_i} \right)^{\frac{1}{\mu_i}} = \left\| u^{\frac{p+m_1}{2}} \right\|_{L^{\frac{2\lambda_i \mu_i}{p+m_1}}(\Omega)}^{\frac{2\lambda_i}{p+m_1}} \\ & \leq C_{318} \left\| \nabla u^{\frac{p+m_1}{2}} \right\|_{L^2(\Omega)}^{\frac{2\lambda_i}{p+m_1} \cdot h_{3i}} \cdot \left\| u^{\frac{p+m_1}{2}} \right\|_{L^{\frac{2\lambda_i \mu_i}{p+m_1}}(\Omega)}^{\frac{2\lambda_i}{p+m_1} \cdot (1-h_{3i})} + C_{318} \left\| u^{\frac{p+m_1}{2}} \right\|_{L^{\frac{2\lambda_i}{p+m_1}}(\Omega)}^{\frac{2\lambda_i}{p+m_1}} \\ & \leq C_{319} \left\| \nabla u^{\frac{p+m_1}{2}} \right\|_{L^2(\Omega)}^{\frac{2\lambda_i}{p+m_1} \cdot h_{3i}} + C_{319} \end{aligned} \quad (3.28)$$

with

$$h_{3i} = \frac{\frac{p+m_1}{2} - \frac{p+m_1}{2\lambda_i\mu_i}}{\frac{p+m_1}{2} - (\frac{1}{2} - \frac{1}{N})} \quad (3.29)$$

and

$$\begin{aligned} \left( \int_{\Omega} |\nabla v|^{2\mu'_i} \right)^{\frac{1}{\mu'_i}} &= \left\| |\nabla v|^q \right\|_{L^{\frac{\kappa_i\mu'_i}{q}}(\Omega)}^{\frac{\kappa_i}{q}} \\ &\leq C_{320} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{\kappa_i}{q} \cdot h_{4i}} \cdot \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{\kappa_i}{q} \cdot (1-h_{4i})} + C_{320} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{\kappa_i}{q}} \\ &\leq C_{321} \left\| \nabla |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{\kappa_i}{q} \cdot h_{4i}} + C_{321} \end{aligned} \quad (3.30)$$

with

$$h_{4i} = \frac{\frac{q}{s} - \frac{q}{\kappa_i\mu'_i}}{\frac{q}{s} - (\frac{1}{2} - \frac{1}{N})}, \quad (3.31)$$

where we have applied the boundedness of  $\|u\|_{L^1}$  and  $\|\nabla v\|_{L^s}$  due to (2.2) and (2.4), (2.9), and  $s \in [1, \frac{N}{N-1})$  if  $0 < \alpha < 1$  and  $s = 2$  if  $\alpha \geq 2$ .

We now claim that for  $p, q$  large there always exist  $\mu_i > 1$ ,  $i = 1, 2$  such that

$$\frac{2\lambda_i\mu_i}{p+m_1} \geq 1, \frac{\kappa_i\mu'_i}{q} \geq 1, 0 < h_{3i}, h_{4i} < 1 \quad (3.32)$$

and under condition (1.4)

$$f_i(p, q, s) := \frac{2\lambda_i}{p+m_1} \cdot h_{3i} + \frac{\kappa_i}{q} \cdot h_{4i} = \frac{\lambda_i - \frac{1}{\mu_i}}{\frac{p+m_1}{2} - (\frac{1}{2} - \frac{1}{N})} + \frac{\frac{\kappa_i}{s} - \frac{1}{\mu'_i}}{\frac{q}{s} - (\frac{1}{2} - \frac{1}{N})} < 2. \quad (3.33)$$

We recall that if  $\alpha + \beta < 2$ , then for any  $\epsilon > 0$ , there exists  $C_{\epsilon} > 0$  such that  $(x^{\alpha} + 1)(y^{\beta} + 1) \leq \epsilon(x^2 + y^2) + C_{\epsilon}$  for all  $x, y > 0$ . Therefore if conditions (3.32) and (3.33) hold, we have

$$\begin{aligned} I_{3i} &\leq \left( \int_{\Omega} |\nabla u|^{\frac{p+m_1}{2}} \right)^{\frac{1}{2} \cdot \frac{2\lambda_i}{p+m_1} \cdot h_{3i}} \cdot \left( \int_{\Omega} \left\| \nabla |\nabla v|^q \right\|^2 \right)^{\frac{1}{2} \cdot \frac{\kappa_i}{q} \cdot (1-h_{3i})} + C_{322} \\ &\leq \epsilon \int_{\Omega} |\nabla u|^{\frac{p+m_1}{2}} + \epsilon \int_{\Omega} \left\| \nabla |\nabla v|^q \right\|^2 + C_{322}. \end{aligned} \quad (3.34)$$

Combining (3.23) with (3.34), we conclude that

$$\frac{d}{dt} \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) + \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) \leq C(p, q) \quad (3.35)$$

for all  $t \in (0, \infty)$ . Then we can apply the Grönwall's lemma to show (3.24).

Now in order to complete the proof of Lemma 3.1, we only need to verify (3.32) and (3.33) claimed above in order to apply the Gagliardo–Nirenberg interpolation inequality. First of all we see that (3.32) is equivalent as

$$\frac{1}{2} - \frac{1}{N} < \frac{p+m_1}{2\lambda_i\mu_i} \leq 1 \text{ and } \frac{1}{2} - \frac{1}{N} < \frac{q}{\kappa_i\mu'_i} \leq 1,$$

which, in terms of (3.26) and (3.27), become

$$\frac{1}{2} - \frac{1}{N} < \frac{p+m_1}{2(p-m_1+2m_2-2)\mu_1} \leq 1, \quad \frac{1}{2} - \frac{1}{N} < \frac{q}{2\mu'_1} \leq 1 \quad (3.36)$$

and

$$\frac{1}{2} - \frac{1}{N} < \frac{p+m_1}{4\mu_2} \leq 1, \quad \frac{1}{2} - \frac{1}{N} < \frac{q}{2(q-1)\mu'_2} \leq 1. \quad (3.37)$$

In the sequel we choose  $\mu_1 := \mu_1(q) = \frac{q}{q-1}$  and  $\mu_2 := \mu_2(p) = \frac{p}{2}$ , and then it is easy to see that (3.36) and (3.37) hold for  $p$  and  $q$  being large.

Finally we are left to prove  $f_i(p, q, s) < 2$  in (3.33) which, in light of (3.26) and (3.27), are

$$f_1(p, q, s) = \frac{p-m_1+2m_2-2-\frac{1}{\mu_1}}{\frac{p+m_1}{2}-\left(\frac{1}{2}-\frac{1}{N}\right)} + \frac{\frac{2}{s}-\frac{1}{\mu'_1}}{\frac{q}{s}-\left(\frac{1}{2}-\frac{1}{N}\right)} < 2$$

and

$$f_2(p, q, s) = \frac{2-\frac{1}{\mu_2}}{\frac{p+m_1}{2}-\left(\frac{1}{2}-\frac{1}{N}\right)} + \frac{\frac{2(q-1)}{s}-\frac{1}{\mu'_2}}{\frac{q}{s}-\left(\frac{1}{2}-\frac{1}{N}\right)} < 2.$$

By straightforward calculations we see that  $f_1(p, q, s) < 2$  and  $f_2(p, q, s) < 2$  are equivalent as

$$\frac{q}{s} > \zeta_1\left(\frac{p+m_1}{2}-\left(\frac{1}{2}-\frac{1}{N}\right)\right) + \left(\frac{1}{2}-\frac{1}{N}\right) \quad (3.38)$$

and

$$\frac{q}{s} < \zeta_2\left(\frac{p+m_1}{2}-\left(\frac{1}{2}-\frac{1}{N}\right)\right) + \left(\frac{1}{2}-\frac{1}{N}\right), \quad (3.39)$$

where

$$\zeta_1 = \zeta_1(p, q, s) := \frac{\frac{1}{s} - \frac{1}{2\mu'_1(p, q)}}{m_1 - m_2 + \frac{1}{2} + \frac{1}{N} + \frac{1}{2\mu_1(p, q)}} > 0$$

and

$$\zeta_2 = \zeta_2(p, q, s) := \frac{\frac{1}{s} + \frac{1}{2\mu'_2(p, q)} - \left(\frac{1}{2} - \frac{1}{N}\right)}{1 - \frac{1}{2\mu_2(p, q)}} > 0.$$

We want to mention that the denominator in  $\zeta_1$  is positive under condition (1.4).

Note that  $\mu_1 = \frac{q}{q-1}$  and  $\mu_2 = \frac{p}{2}$  and then our discussions are divided into the followings:  
case (i).  $0 < \alpha < 1$  and therefore  $s \in [1, \frac{N}{N-1})$ . Then  $m_2 - m_1 < \frac{2}{N}$  implies

$$\zeta_2\left(\infty, \infty, \frac{N}{N-1}\right) - \zeta_1\left(\infty, \infty, \frac{N}{N-1}\right) = 1 - \frac{1 - \frac{1}{N}}{m_1 - m_2 + 1 + \frac{1}{N}} = \frac{m_1 - m_2 + \frac{2}{N}}{m_1 - m_2 + 1 + \frac{1}{N}} > 0.$$

By the continuity of  $\zeta_i$ , for all  $p, q$  sufficiently large and  $s$  smaller than but close to  $\frac{N}{N-1}$ , we have that  $\zeta_2(p, q, s) > \zeta_1(p, q, s)$  and therefore both (3.38) and (3.39) hold for such  $(p, q, s)$  hence  $f_i(p, q, s) < 2$ .

case (ii).  $\alpha \geq 1$  and therefore  $s = 2$ . Then  $m_2 - m_1 < \frac{3N+2}{N(N+2)}$  implies

$$\zeta_2(\infty, \infty, 2) - \zeta_1(\infty, \infty, 2) = \left(\frac{1}{2} + \frac{1}{N}\right) - \frac{\frac{1}{2}}{m_1 - m_2 + 1 + \frac{1}{N}} = \frac{m_1 - m_2 + \frac{3N+2}{N(N+2)}}{(m_1 - m_2 + 1 + \frac{1}{N})\left(\frac{1}{2} + \frac{1}{N}\right)} > 0.$$

Similar as in case (i) we have that  $\zeta_2(p, q, 2) > \zeta_1(p, q, 2)$  hence  $f_i(p, q, 2) < 2$  when  $p, q$  are large. In both cases (3.33) holds for large  $p, q$  under condition (1.4) and this completes the proof of Lemma 3.1.  $\square$

In the following lemma, we estimate  $I_{31}$  and  $I_{32}$  by using Young's inequality instead of Hölder's as in Lemma 3.1. We shall see that  $\alpha$  plays an important role in *a priori* estimates.

**Lemma 3.2.** *Suppose that*

$$2m_2 - m_1 < \begin{cases} \max\{\alpha, m_1\} + \frac{2}{N}, & \text{if } 0 < \alpha < 1, \\ \max\{\alpha, m_1\} + \frac{4}{N+2}, & \text{if } \alpha \geq 1, \end{cases} \quad (3.40)$$

*then for large  $p$  and  $q$  there exists a constant  $C(p, q) > 0$  such that*

$$\int_{\Omega} u^p + \int_{\Omega} |\nabla v|^{2q} \leq C(p, q), \forall t \in (0, \infty). \quad (3.41)$$

*Proof.* First of all, we invoke the Gagliardo–Nirenberg interpolation inequality

$$\begin{aligned} \int_{\Omega} u^{p+m_1} &= \|u^{\frac{p+m_1}{2}}\|_{L^2(\Omega)}^2 \\ &\leq C_{323} \|\nabla u^{\frac{p+m_1}{2}}\|_{L^2(\Omega)}^{2h_5} \cdot \|u^{\frac{p+m_1}{2}}\|_{L^{\frac{2}{p+m_1}}(\Omega)}^{2(1-h_5)} + C_{323} \|u^{\frac{p+m_1}{2}}\|_{L^{\frac{2}{p+m_1}}(\Omega)}^2 \\ &\leq C_{324} \|\nabla u^{\frac{p+m_1}{2}}\|_{L^2(\Omega)}^{2h_5} + C_{324}, \end{aligned} \quad (3.42)$$

where we have applied the fact that  $\|u\|_{L^1}$  is bounded and

$$h_5 := \frac{\frac{p+m_1}{2} - \frac{1}{2}}{\frac{p+m_1}{2} - (\frac{1}{2} - \frac{1}{N})} \in (0, 1).$$

By Young's inequality, there exists a positive constant  $C_{323}$  such that in (3.23)

$$\begin{aligned} \frac{\chi^2 M_2^2 (p-1)}{2M_1} I_{31} &\leq \frac{b_1}{4} \int_{\Omega} (u^{p-m_1+2m_2-2})^{\frac{p+\max\{\alpha, m_1\}}{p-m_1+2m_2-2}} + C_{325} \int_{\Omega} |\nabla v|^{2 \cdot \frac{p+\max\{\alpha, m_1\}}{\max\{\alpha, m_1\}+m_1-2m_2+2}} \\ &= \frac{b_1}{4} \int_{\Omega} u^{p+\max\{\alpha, m_1\}} + C_{325} \int_{\Omega} |\nabla v|^{\theta_1} \end{aligned} \quad (3.43)$$

and

$$\begin{aligned} C_{314} I_{32} &\leq \frac{b_1}{4} \int_{\Omega} (u^2)^{\frac{p+\max\{\alpha, m_1\}}{2}} + C_{326} \int_{\Omega} |\nabla v|^{2(q-1) \cdot \frac{p+\max\{\alpha, m_1\}}{p+\max\{\alpha, m_1\}-2}} \\ &= \frac{b_1}{4} \int_{\Omega} u^{p+\max\{\alpha, m_1\}} + C_{326} \int_{\Omega} |\nabla v|^{\theta_2}, \end{aligned} \quad (3.44)$$

where we denote

$$\theta_1 := \theta_1(p, q) = \frac{2(p + \max\{\alpha, m_1\})}{\max\{\alpha, m_1\} + m_1 - 2m_2 + 2} \quad (3.45)$$

and

$$\theta_2 := \theta_2(p, q) = \frac{2(q-1)(p + \max\{\alpha, m_1\})}{p + \max\{\alpha, m_1\} - 2}. \quad (3.46)$$

We want to mention that  $\theta_i$  are well-defined since  $\max\{\alpha, m_1\} > 2m_2 - m_1 - 2$  thanks to (3.40). Substituting (3.43)–(3.46) into (3.23), we derive

$$\begin{aligned} &\frac{d}{dt} \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) + \left( \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q} \right) + \frac{(q-1)D_2}{4q^2} \int_{\Omega} |\nabla |\nabla v|^q|^2 \\ &\leq C_{325} \int_{\Omega} |\nabla v|^{\theta_1} + C_{326} \int_{\Omega} |\nabla v|^{\theta_2} + C_{327}. \end{aligned} \quad (3.47)$$

According to Gagliardo–Nirenberg interpolation inequality, we have for  $i = 1, 2$

$$\begin{aligned} \int_{\Omega} |\nabla v|^{\theta_i} &= \left\| |\nabla v|^q \right\|_{L^{\frac{\theta_i}{q}}(\Omega)}^{\frac{\theta_i}{q}} \\ &\leq C_{328} \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{\theta_i}{q} h_{6i}} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{\theta_i}{q} (1-h_{6i})} + C_{328} \left\| |\nabla v|^q \right\|_{L^{\frac{s}{q}}(\Omega)}^{\frac{\theta_i}{q}} \\ &\leq C_{329} \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^{\frac{\theta_i}{q} h_{6i}} + C_{330}, \end{aligned}$$

where we have applied the boundedness of  $\|\nabla v\|_{L^s(\Omega)}$  and

$$h_{6i} := h_{6i}(p, q; s) = \frac{\frac{q}{s} - \frac{q}{\theta_i}}{\frac{q}{s} - (\frac{1}{2} - \frac{1}{N})}.$$

Denote

$$g_i(p, q; s) := \frac{\theta_i}{q} h_{6i}(p, q; s),$$

and then we want to claim that under condition (3.40) there exists  $p$  and  $q$  large such that the followings hold

$$0 < h_{6i}(p, q; s) < 1 \text{ and } 0 < g_i(p, q; s) < 2. \quad (3.48)$$

Assuming (3.48), we conclude from Gagliardo–Nirenberg interpolation inequality and the Young's inequality that for any  $\epsilon > 0$

$$\int_{\Omega} |\nabla v|^{\theta_i} \leq \epsilon \left\| |\nabla v|^q \right\|_{L^2(\Omega)}^2 + C_{\epsilon}. \quad (3.49)$$

Substituting (3.49) into (3.47), we can easily derive that

$$y'(t) + y(t) \leq C_{325},$$

by setting  $y(t) := \frac{1}{p} \int_{\Omega} u^p + \frac{1}{2q} \int_{\Omega} |\nabla v|^{2q}$  and solving this inequality by Grönwall's lemma gives rise to (3.41).

Now we need to verify the inequalities in (3.48), which by straightforward calculations, are equivalent as

$$\theta_i > s, q > \frac{\theta_i}{2} - \frac{s}{N}.$$

It is easy to see that  $\theta_i > s$  hold since both  $p$  and  $q$  chosen to be large, and therefore we shall only need to verify that  $q > \frac{\theta_i}{2} - \frac{s}{N}$  in the sequel. We divide our discussions into the following two cases: case (i).  $0 < \alpha < 1$  and therefore  $s \in [1, \frac{N}{N-1})$ . Then we can solve the inequalities  $q > \frac{\theta_i}{2} - \frac{s}{N}$  for  $i = 1, 2$  to see that

$$q_1(p) < q < q_2(p), \quad (3.50)$$

with

$$q_1(p) = \frac{p + \max\{\alpha, m_1\}}{\max\{\alpha, m_1\} + m_1 - 2m_2 + 2} - \frac{s}{N}$$

and

$$q_2(p) = \frac{(N+s)(p + \max\{\alpha, m_1\})}{2N} - \frac{s}{N}.$$

If  $2m_2 - m_1 < \max\{\alpha, m_1\} + \frac{2}{N}$  in (3.40) holds, we can always find  $s$  smaller than but sufficiently close to  $\frac{N}{N-1}$  such that  $\frac{1}{\max\{\alpha, m_1\} + m_1 - 2m_2 + 2} < \frac{N+s}{2N}$  hence (3.50) holds for  $p, q$  being large.

case (ii).  $\alpha \geq 1$  and therefore  $s = 2$ . The arguments are the same as in case (i) except that now the condition  $q_1(p) < q < q_2(p)$ , which implies that  $\frac{1}{\max\{\alpha, m_1\} + m_1 - 2m_2 + 2} < \frac{N+2}{2N}$ , holds provided that  $2m_2 - m_1 < \max\{\alpha, m_1\} + \frac{4}{N+2}$ .

Therefore in both cases we have verified (3.48) for  $p$  and  $q$  large under (3.40) and the proof of Lemma 3.2 completes.  $\square$

### 3.2 Global existence and boundedness

*Proof of Theorem 1.1.* Taking some  $p > N$  fixed, we have from Lemma 2.1 and Lemma 3.1 that  $\|v(\cdot, t)\|_{W^{1,\infty}}$  is uniformly bounded. Then one can apply the standard Moser–Alikakos  $L^p$  iteration [1] or the user–friendly version in Lemma A.1 of [46] to establish the uniform boundedness of  $\|u(\cdot, t)\|_{L^\infty}$  for (1.1). Therefore the local solution  $(u, v)$  is global thanks to the extension criterion in Proposition 1. Finally, we can apply the standard parabolic regularity theory to show that  $(u, v)$  has the regularity in the theorem.  $\square$

*Proof of Theorem 1.2.* The proof is the same as that of Theorem 1.1 in light of Lemma 3.2.  $\square$

## 4 Parabolic–elliptic system in multi–dimensional domain

In this section, we prove the global existence and boundedness of the classical solutions to the parabolic–elliptic system of (1.1). This model describes a competition relationship that  $v$  diffuses much faster than  $u$ .

### 4.1 Parabolic–elliptic system with repulsion

First of all, we consider the parabolic–elliptic system of (1.1) of the following form

$$\begin{cases} u_t = \nabla \cdot (D_1(u) \nabla u + \chi \phi(u) \nabla v) + (a_1 - b_1 u^\alpha - c_1 v)u, & x \in \Omega, t > 0, \\ 0 = D_2 \Delta v + (a_2 - b_2 u - c_2 v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \quad (4.1)$$

Our first result concerning (4.1) is the following Theorem.

**Theorem 4.1.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . Assume that the smooth functions  $D_1(u)$  and  $\phi(u)$  satisfy (1.2) and (1.3) respectively with*

$$2m_2 - m_1 < \max\{\alpha, m_1\} + 1. \quad (4.2)$$

*Suppose that  $u_0 \in C^0(\bar{\Omega})$  and  $u_0 > 0$  in  $\Omega$ . Then (4.1) admits a unique positive classical solution  $(u, v)$  which is uniformly bounded in  $\Omega \times (0, \infty)$ .*

*Proof.* The proof is very similar as that of Theorem 1.1. First of all, the local existence in  $\Omega \times (0, T_{\max})$  follows from the theory Amann in [2]. Moreover one can easily apply maximum principle and Hopf’s lemma to show that  $u(x, t) \geq, \neq 0$  in  $\Omega \times (0, \infty)$  and  $0 < v(x) < \frac{a_2}{c_2}$  in  $\Omega$ . Furthermore, if  $\|\nabla u(\cdot, t)\|_{L^p}$  is bounded for some  $p > N$ , then  $\|\nabla v\|_{L^\infty}$  is also bounded after applying the Sobolev embedding  $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$  to the  $v$ –equation, and therefore one can apply the standard Moser–Alikakos  $L^p$ –iteration to establish the boundedness of  $\|u\|_{L^\infty}$ . Finally the regularity of  $(u, v)$  follows from parabolic and elliptic embedding theory.



Now we only need to prove the boundedness of  $\int_{\Omega} u^p(\cdot, t)$  for some  $p > N$ . Testing the  $u$ -equation in (4.1) by  $u^{p-1}$  and then integrating it over  $\Omega$  by parts, we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &= -(p-1) \int_{\Omega} D_1(u) u^{p-2} |\nabla u|^2 - (p-1) \int_{\Omega} \chi \phi(u) u^{p-2} \nabla u \nabla v \\ &\quad + \int_{\Omega} u^p (a_1 - b_1 u^\alpha - c_1 v). \end{aligned} \quad (4.3)$$

Similar as in (3.2), we invoke the Gagliardo–Nirenberg interpolation inequality to obtain

$$\begin{aligned} -(p-1) \int_{\Omega} D_1(u) u^{p-2} |\nabla u|^2 &\leq -\frac{4M_1(p-1)}{(p+m_1)^2} \int_{\Omega} |\nabla u^{\frac{p+m_1}{2}}|^2 + C_{41} \\ &\leq -\xi \int_{\Omega} u^{p+m_1} + C_{42}(\xi), \end{aligned} \quad (4.4)$$

and apply Young's inequality to have that for any  $\gamma > 2$

$$\begin{aligned} &-(p-1) \int_{\Omega} \chi \phi(u) u^{p-2} \nabla u \nabla v \\ &\leq \epsilon \int_{\Omega} u^{p+m_1-2} |\nabla u|^2 + \epsilon \int_{\Omega} u^{\frac{(p-m_1+2m_2-2)}{2} \cdot \frac{2\gamma}{\gamma-2}} + C_{\epsilon} \int_{\Omega} |\nabla v|^{\gamma}, \end{aligned} \quad (4.5)$$

where in (4.4) and (4.5)  $\xi > 0$  is arbitrary and  $C_{41}, C_{42}$  are positive constants.

We invoke the Gagliardo–Nirenberg interpolation inequality and the boundedness of  $v$  to obtain

$$\begin{aligned} \int_{\Omega} |\nabla v|^{\gamma} &= \|\nabla v\|_{L^{\gamma}(\Omega)}^{\gamma} \leq C_{43} \|\Delta v\|_{L^{\frac{\gamma}{2}}(\Omega)}^{\frac{\gamma}{2}} \cdot \|v\|_{L^{\infty}(\Omega)}^{\frac{\gamma}{2}} + C_{44} \|\nabla v\|_{L^{\infty}(\Omega)}^{\gamma} \\ &\leq C_{45} \|\Delta v\|_{L^{\frac{\gamma}{2}}(\Omega)}^{\frac{\gamma}{2}} + C_{45}, \end{aligned} \quad (4.6)$$

where  $C_{4i}$  are positive constants. Furthermore, in light of  $D_2 \Delta v = -(a_2 - b_2 u - c_2 v)v$  and the boundedness of  $v$ , (4.6) implies

$$\int_{\Omega} |\nabla v|^{\gamma} = \|\nabla v\|_{L^{\gamma}(\Omega)}^{\gamma} \leq C_{46} \|u\|_{L^{\frac{\gamma}{2}}(\Omega)}^{\frac{\gamma}{2}} + C_{47} = C_{46} \int_{\Omega} u^{\frac{\gamma}{2}} + C_{47}. \quad (4.7)$$

Thanks to (4.4), (4.5) and (4.7), we derive from (4.3)

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq -\xi \int_{\Omega} u^{p+m_1} - b_1 \int_{\Omega} u^{p+\alpha} + \epsilon \int_{\Omega} u^{\frac{(p-m_1+2m_2-2)\gamma}{\gamma-2}} + C_{\epsilon} C_{46} \int_{\Omega} u^{\frac{\gamma}{2}} + C_{\epsilon} C_{47}. \quad (4.8)$$

Choosing  $\gamma = 2(p - m_1 + 2m_2 - 1)$  with  $\frac{\gamma}{2} = \frac{(p-m_1+2m_2-2)\gamma}{\gamma-2}$ , we infer from (4.8)

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &\leq -\xi \int_{\Omega} u^{p+m_1} - b_1 \int_{\Omega} u^{p+\alpha} + (\epsilon + C_{\epsilon} C_{46}) \int_{\Omega} u^{p-m_1+2m_2-1} + C_{\epsilon} C_{47} \\ &\leq -\int_{\Omega} u^p + C_{48}, \end{aligned} \quad (4.9)$$

where the second inequality follows from (4.2). Solving (4.9) implies that  $\|u(\cdot, t)\|_{L^p}$  is uniformly bounded in time for each  $p \geq 2$  and this completes the proof.  $\square$

**Remark 1.** If  $m_1 \geq \alpha$ , then Theorem 4.1 also holds if (4.2) is relaxed to  $2m_2 - m_1 \leq \max\{\alpha, m_1\} + 1$  or equivalently  $2m_2 - 2m_1 \leq 1$ . Indeed, in this case we can choose  $\xi > 2(\epsilon + C_{\epsilon} C_{46})$  and therefore (4.9) implies that

$$\frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p \leq -\frac{\xi}{2} \int_{\Omega} u^{p+m_1} + C_{48},$$

from which the boundedness of  $\int_{\Omega} u^p(x, t)$  follows. Similarly one can show that Theorem 4.1 holds for  $2m_2 - m_1 \leq \max\{\alpha, m_1\} + 1$  when  $m_1 < \alpha$  and  $b_1$  is large.

By a different approach we prove the following results.

**Theorem 4.2.** Suppose that all the conditions in Theorem 4.1 hold except that

$$m_2 < \max\{\alpha, m_1\} \quad (4.10)$$

then the nonnegative solution  $(u, v)$  to (4.1) is classical and bounded in  $\Omega \times (0, \infty)$ .

*Proof.* We begin with (4.3) and estimate the second term differently. For each  $p > 2$  we denote

$$\Phi_p(u) = \int_0^u \phi(s) s^{p-2} ds,$$

then thanks to  $\phi(s) \leq M_2 s^{m_2}$

$$\Phi_p(u) \leq M_2 \int_0^u s^{p+m_2-2} ds = \frac{M_2}{p+m_2-1} u^{p+m_2-1}.$$

Therefore we have from the integration by parts and the second equation in (4.1)

$$\begin{aligned} & - (p-1) \int_{\Omega} \chi \phi(u) u^{p-2} \nabla u \nabla v \\ &= - (p-1) \chi \int_{\Omega} \nabla \Phi_p(u) \nabla v = (p-1) \chi \int_{\Omega} \Phi_p(u) \Delta v \\ &= - (p-1) \chi \int_{\Omega} \Phi_p(u) (a_2 - b_2 u - c_2 v) v \\ &= b_2 (p-1) \chi \int_{\Omega} \Phi_p(u) u v + (p-1) \chi \int_{\Omega} \Phi_p(u) (c_2 v - a_2) v \\ &\leq \frac{b_2 (p-1) \chi M_2 \|v\|_{L^\infty}}{p+m_2-1} \int_{\Omega} u^{p+m_2} + \frac{b_2 (p-1) \chi M_2 \|(c_2 v - a_2) v\|_{L^\infty}}{p+m_2-1} \int_{\Omega} u^{p+m_2-1} \\ &\leq C_{49} \int_{\Omega} u^{p+m_2} + C_{410}, \end{aligned} \quad (4.11)$$

where  $C_{49}$  and  $C_{410}$  are positive constants.

Collecting (4.4) and (4.11) we have from (4.3) that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &\leq -\xi \int_{\Omega} u^{p+m_1} - b_1 \int_{\Omega} u^{p+\alpha} + C_{410} \int_{\Omega} u^{p+m_2} + C_{412} \\ &\leq - \int_{\Omega} u^p + C_{411}, \end{aligned} \quad (4.12)$$

where the second inequality follows from the fact that  $m_2 < \max\{\alpha, m_1\}$ . This implies the boundedness of  $\int_{\Omega} u^p$  for each  $p > 2$  and the rest proof is the same as that of Theorem 4.1.  $\square$

**Remark 2.** Similar as Remark 1, one can show that (4.10) can be relaxed to  $m_2 \leq \max\{\alpha, m_1\}$  if  $m_1 > \alpha_1$  or  $b_1$  is large.

## 4.2 Parabolic–elliptic system with attraction

Finally, we establish the global existence and boundedness of the following parabolic–elliptic system with attraction

$$\begin{cases} u_t = \nabla \cdot (D_1(u)\nabla u - \chi\phi(u)\nabla v) + (a_1 - b_1u^\alpha - c_1v)u, & x \in \Omega, t > 0, \\ 0 = D_2\Delta v + (a_2 - b_2u - c_2v)v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial \mathbf{n}} = \frac{\partial v}{\partial \mathbf{n}} = 0, & x \in \partial\Omega, t > 0, \\ u(x, 0) = u_0(x) \geq 0, & x \in \Omega. \end{cases} \quad (4.13)$$

Here  $D_1(u) \geq M_1(1 + u)^{m_1}$  as in (1.1) while condition (1.3) changes to  $\phi(u) \geq M_2u^{m_2}$ . We prove global existence and boundedness for (4.13) for any  $m_i > 0$  and  $\alpha$ . The last main result of this paper is the following theorem.

**Theorem 4.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$ ,  $N \geq 1$ , with piecewise smooth boundary. Assume that  $u_0 \in C^0(\bar{\Omega})$  and  $u_0 > 0$  in  $\Omega$ . Suppose for some positive constants  $M_i > 0$ ,  $D_1(u) \geq M_1(1 + u)^{m_1}$  and  $\phi(u) \geq M_2u^{m_2}$ , with  $\max\{m_1, m_2, \alpha\} \geq 0$ . Then (4.13) has a unique positive solution  $(u, v)$  which is classical and uniformly bounded in  $\Omega \times (0, \infty)$ .*

*Proof.* By the same arguments for (4.3) we test the  $u$ -equation in (4.13) by  $u^{p-1}$  and integrate it over  $\Omega$  by parts to obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &= - (p-1) \int_{\Omega} D_1(u) u^{p-2} |\nabla u|^2 + (p-1) \int_{\Omega} \chi\phi(u) u^{p-2} \nabla u \nabla v \\ &\quad + \int_{\Omega} u^p (a_1 - b_1u^\alpha - c_1v). \end{aligned} \quad (4.14)$$

Similar as above, we denote

$$\tilde{\Phi}_p(u) = \int_0^u \phi(s) s^{p-2} ds,$$

and then we can show  $\tilde{\Phi}_p(u) \geq \frac{M_2}{p+m_2-1} u^{p+m_2-1}$  and derive as in (4.11)

$$\begin{aligned} &(p-1) \int_{\Omega} \chi\phi(u) u^{p-2} \nabla u \nabla v \\ &= -b_2(p-1) \chi \int_{\Omega} \tilde{\Phi}_p(u) uv + (p-1) \chi \int_{\Omega} \tilde{\Phi}_p(u) (c_2v - a_2)v \\ &\leq -C_{412} \int_{\Omega} u^{p+m_2} + C_{413}, \end{aligned} \quad (4.15)$$

where  $C_{412}$  and  $C_{413}$  are positive constants. Collecting (4.15) and (4.4), we have from (4.14)

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \int_{\Omega} u^p &\leq -\xi \int_{\Omega} u^{p+m_1} - b_1 \int_{\Omega} u^{p+\alpha} - C_{412} \int_{\Omega} u^{p+m_2} + C_{413} \\ &\leq - \int_{\Omega} u^p + C_{414}, \end{aligned} \quad (4.16)$$

where  $C_{414}$  is a positive constant and the last inequality follows from Young's inequality and the assumption that  $\max\{m_1, m_2, \alpha\} \geq 0$ . Solving (4.16) gives rise to the boundedness of  $\int_{\Omega} u^p$  for any  $p > 2$  hence the global existence and boundedness follow.  $\square$

According to Theorem 4.3, only one of  $m_1$ ,  $m_2$  and  $\alpha$  needs to be nonnegative to guarantee the global existence and boundedness of (4.13) in contrast to Theorem 4.1 and Theorem 4.2. Apparently this is due to the effect of population attraction. It is necessary to point out that for

chemotaxis model, it is well known that chemo–attraction destabilizes the system and supports the occurrence of blowups, while chemo–repulsion tends to prevent blowups. However, for Lotka–Volterra competition models, attraction prevents blowups while repulsion, though not completely understood, tends to support blowups according to Theorem 4.3. See [27, 51] for instance. We surmise that the same conclusions hold true for the fully parabolic system (1.1) and a completely different approach is needed to show this.

## References

- [1] N. D. Alikakos,  *$L^p$  bounds of solutions of reaction–diffusion equations*, Comm. Partial Differential Equations, **4** (1979), 827–868.
- [2] H. Amann, *Nonhomogeneous linear and quasilinear elliptic and parabolic boundary value problems*, Function Spaces, differential operators and nonlinear Analysis, Teubner, Stuttgart, Leipzig, **133** (1993), 9–126.
- [3] M. Burger, M. Di Francesco and Y. Dolak-Struss, *The Keller–Segel model for chemotaxis with prevention of overcrowding: linear vs. nonlinear diffusion*, SIAM J. Math. Anal., **38** (2006), 1288–1315.
- [4] V. Calvez and J. A. Carrillo, *Volume effects in the Keller–Segel model: energy estimates preventing blow-up*, J. Math. Pures Appl., **86** (2006), 155–175.
- [5] R. Cantrell, C. Cosner and Y. Lou, *Approximating the ideal free distribution via reaction–diffusion–advection equations*, J. Differential Equations, **245** (2008), 3687–3703.
- [6] R. Cantrell, C. Cosner, Y. Lou and C. Xie, *Random dispersal versus fitness–dependent dispersal*, J. Differential Equations, **254** (2013), 2905–2941.
- [7] L. Chen and A. Jüngel, *Analysis of a multi-dimensional parabolic population model with strong cross–diffusion*, SIAM J. Math. Anal., **36** (2004), 301–322.
- [8] L. Chen and A. Jüngel, *Analysis of a parabolic cross–diffusion population model without self–diffusion*, J. Differential Equations, **224** (2006), 39–59.
- [9] Y. S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada–Kawasaki–Teramoto model with weak cross–diffusion*, Discrete Contin. Dyn. Syst., **9** (2003), 1193–1200.
- [10] Y. S. Choi, R. Lui and Y. Yamada, *Existence of global solutions for the Shigesada–Kawasaki–Teramoto model with strongly coupled cross–diffusion*, Discrete Contin. Dyn. Syst., **10** (2004), 719–730.
- [11] T. Cieślak and C. Stinner, *New critical exponents in a fully parabolic quasilinear Keller–Segel system and applications to volume filling models*, J. Differential Equations, **258** (2015), 2080–2113.
- [12] E. Conway, D. Hoff and J. Smoller, *Large time behavior of solutions of systems of nonlinear reaction–diffusion equations*, SIAM J. Appl. Math., **35**, (1978), 1–16.
- [13] E. Conway and J. Smoller, *A comparison technique for systems of reaction–diffusion equations*, Comm. Partial Differential Equations, **2** (1977), 679–697.
- [14] C. Cosner, *Reaction–diffusion–advection models for the effects and evolution of dispersal*, Discrete Contin. Dyn. Syst., **34** (2014), 1701–1745.
- [15] P. De Mottoni and F. Rothe, *Convergence to homogeneous equilibrium state for generalized Volterra–Lotka systems with diffusion*, SIAM J. Appl. Math., **37** (1979), 648–663.
- [16] L. Desvillettes, T. Lepoutre, A. Moussa and A. Trescases, *On the entropic structure of reaction–cross diffusion systems*, Comm. Partial Differential Equations, **40** (2015), 1705–1747.

- [17] L. Hoang, T. Nguyen and T. Phan, *Gradient estimates and global existence of smooth solutions to a cross-diffusion system*, SIAM J. Math. Anal., **47** (2015), 2122–2177.
- [18] S. Ishida, K. Seki and T. Yokota, *Boundedness in quasilinear Keller–Segel systems of parabolic–parabolic type on non-convex bounded domains*, J. Differential Equations, **256** (2014), 2993–3010.
- [19] A. Jüngel, *The boundedness-by-entropy principle for cross-diffusion systems*, Nonlinearity, **28** (2015), 1963–2001.
- [20] K. Kishimoto and H. Weinberger, *The spatial homogeneity of stable equilibria of some reaction–diffusion systems in convex domains*, J. Differential Equations, **58** (1985), 15–21.
- [21] T. Kolokolnikov and J. Wei, *Stability of spiky solutions in a competition model with cross-diffusion*, SIAM J. Appl. Math., **71** (2011), 1428–1457.
- [22] K. Kuto, *Limiting structure of shrinking solutions to the stationary Shigesada–Kawasaki–Teramoto model with large cross-diffusion*, SIAM J. Math. Anal., **47** (2015), 3993–4024.
- [23] K. Kuto and T. Tsujikawa, *Limiting structure of steady-states to the Lotka–Volterra competition model with large diffusion and advection*, J. Differential Equations, **258** (2015), 1801–1858.
- [24] J. Lankeit, *Chemotaxis can prevent thresholds on population density*, Discrete Contin. Dyn. Syst., Ser. B, **20** (2015), 1499–1527.
- [25] D. Le, *Cross diffusion systems on  $n$  spatial dimensional domains*, Indiana Univ. Math. J., **51** (2002), 625–643.
- [26] D. Le and V. Nguyen, *Global solutions to cross diffusion parabolic systems on 2D domains*, Proc. Amer. Math. Soc., **143** (2015), 2999–3010.
- [27] D. Le and V. Nguyen, *Global and blow up solutions to cross diffusion systems on 3D domains*, Proc. Amer. Math. Soc., (2016), published online.
- [28] D. Le, L. Nguyen and T. Nguyen, *Coexistence in cross diffusion systems*, Indiana Univ. Math. J., **56** (2007), 1749–1791.
- [29] Y. Lou and W.-M. Ni, *Diffusion, self-diffusion and cross-diffusion*, J. Differential Equations, **131** (1996), 79–131.
- [30] Y. Lou and W.-M. Ni, *Diffusion vs cross-diffusion: An elliptic approach*, J. Differential Equations, **154** (1999), 157–190.
- [31] Y. Lou, W.-M. Ni and Y. Wu, *On the global existence of a cross-diffusion system*, Discrete Contin. Dynam. Systems, **4** (1998), 193–203.
- [32] Y. Lou, W.-M. Ni and S. Yotsutani, *On a limiting system in the Lotka–Volterra competition with cross-diffusion*, Discrete Contin. Dyn. Syst., **10** (2004), 435–458.
- [33] Y. Lou, W.-M. Ni and S. Yotsutani, *Pattern formation in a cross-diffusion system*, Discrete Contin. Dyn. Syst., **35** (2015), 1589–1607.
- [34] Y. Lou and M. Winkler, *Global existence and uniform boundedness of smooth solutions to a cross-diffusion system with equal diffusion rates*, Comm. Partial Differential Equations, **40** (2015), 1905–1941.
- [35] Y. Lou, M. Winkler and Y. Tao, *Approaching the ideal free distribution in two-species competition models with fitness-dependent dispersal*, SIAM J. Math. Anal., **46** (2014), 1228–1262.
- [36] H. Matano and M. Mimura, *Pattern formation in competition–diffusion systems in non-convex domains*, Publ. Res. Inst. Math. Sci., **19** (1983), 1049–1079.
- [37] M. Mimura, S.-I. Ei and Q. Fang, *Effect of domain-shape on coexistence problems in a competition–diffusion system*, J. Math. Biol., **29** (1991), 219–237.

- [38] M. Mimura and K. Kawasaki, *Spatial segregation in competitive interaction-diffusion equations*, J. Math. Biol., **9** (1980), 49–64.
- [39] M. Mimura, Y. Nishiura, A. Tesei and T. Tsujikawa, *Coexistence problem for two competing species models with density-dependent diffusion*, Hiroshima Math. J., **14** (1984), 425–449.
- [40] W.-M. Ni, Y. Wu and Q. Xu, *The existence and stability of nontrivial steady states for S-K-T competition model with cross diffusion*, Discrete Contin. Dyn. Syst., **34** (2014), 5271–5298.
- [41] L. Shao, Y. Song and Q. Wang, *Boundedness and persistence of populations in advective Lotka-Volterra competition system*, preprint.
- [42] N. Shigesada, K. Kawasaki and E. Teramoto, *Spatial segregation of interacting species*, J. Theoret. Biol., **79** (1979), 83–99.
- [43] S.-A. Shim, *Uniform boundedness and convergence of solutions to cross-diffusion systems*, J. Differential Equations, **185** (2002), 281–305.
- [44] Y. Sugiyama and H. Kunii, *Global existence and decay properties for a degenerate Keller-Segel model with a power factor in drift term*, J. Differential Equations, **227** (2006), 333–364.
- [45] Y. Tao and M. Winkler, *A chemotaxis-haptotaxis model: the roles of nonlinear diffusion and logistic source*, SIAM J. Math. Anal., **43** (2011), 685–704.
- [46] Y. Tao and M. Winkler, *Boundedness in a quasilinear parabolic-parabolic Keller-Segel system with subcritical sensitivity*, J. Differential Equations, **252** (2012), 692–715.
- [47] P. Tuoc, *Global existence of solutions to Shigesada-Kawasaki-Teramoto cross-diffusion systems on domains of arbitrary dimensions*, Proc. Amer. Math. Soc., **135** (2007), 3933–3941.
- [48] P. V. Tuoc and V. Phan, *On global existence of solutions to a cross-diffusion system*, J. Math. Anal. Appl., **343** (2008), 826–834.
- [49] L. Wang, C. Mu and S. Zhou, *Boundedness in a parabolic-parabolic chemotaxis system with nonlinear diffusion*, Z. Angew. Math. Phys., **65** (2014), 1137–1152.
- [50] Q. Wang, *On the steady state of a shadow system to the SKT competition model*, Discrete Contin. Dyn. Syst.-Series B, **19** (2014), 2941–2961.
- [51] Q. Wang, C. Gai and J. Yan, *Qualitative analysis of a Lotka-Volterra competition system with advection*, Discrete Contin. Dyn. Syst., **35** (2015), 1239–1284.
- [52] Q. Wang and L. Zhang, *On the multi-dimensional advective Lotka-Volterra competition systems*, preprint.
- [53] Y. Wang, *Boundedness in the higher-dimensional chemotaxis-haptotaxis model with nonlinear diffusion*, J. Differential Equations, **260** (2016), 1975–1989.
- [54] M. Winkler, *Aggregation vs. global diffusive behavior in the higher-dimensional Keller-Segel model*, J. Differential Equations, **248** (2010), 2889–2905.
- [55] M. Winkler, *Boundedness in the higher-dimensional parabolic-parabolic chemotaxis system with logistic source*, Comm. Partial Differential Equations, **35** (2010), 1516–1537.
- [56] M. Winkler, *Blow-up in a higher-dimensional chemotaxis system despite logistic growth restriction*, J. Math. Anal. Appl., **384** (2011), 261–272.
- [57] M. Winkler, *How far can chemotactic cross-diffusion enforce exceeding carrying capacities?*, J. Nonlinear Sci., **24** (2014), 809–855.
- [58] Y. Wu and Q. Xu, *The existence and structure of large spiky steady states for SKT competition systems with cross-diffusion*, Discrete Contin. Dyn. Syst., **29** (2011), 367–385.
- [59] Y. Yamada, *Global solutions for the Shigesada-Kawasaki-Teramoto model with cross-diffusion*, Recent progress on reaction-diffusion systems and viscosity solutions, World Scientific River Edge, NJ, 2009, 282–299.

- [60] Q. Zhang and Y. Li, *Boundedness in a quasilinear fully parabolic Keller–Segel system with logistic source*, Z. Angew. Math. Phys., **66** (2015), 2473–2484.
- [61] P. Zheng, C. Mu and X. Hu, *Boundedness and blow-up for a chemotaxis system with generalized volume-filling effect and logistic source*, Discrete Contin. Dyn. Syst., **35** (2015), 2299–2323.
- [62] P. Zheng, C. Mu and X. Song, *On the boundedness and decay of solutions for a chemotaxis–haptotaxis system with nonlinear diffusion*, Discrete Contin. Dyn. Syst., **36** (2016), 1737–1757.